SOLUTIONS OF PROBLEMS.

ALGEBRA.

408. Proposed by EMMA M. GIBSON, Drury College, Mo.

Show that if n is a positive integer, the sum of the series

$$1-\frac{2n-1}{\underline{|1}}+\frac{(2n-1)(2n-2)}{\underline{|2}}-\cdots+(-1)^{n-1}\frac{(2n-1)(2n-2)\cdots(n+1)}{\underline{|n-1}}$$

is

$$\frac{(-1)^{n-1}(2n-2)(2n-3)\cdots(n+1)n}{\underline{|n-1}}.$$

Solution by Howard C. Feemster, York College, Neb.

Let

$$S_1=1,\qquad S_2=-\frac{2n-2}{\underline{|1}},\qquad S_3=\frac{(2n-2)(2n-3)}{\underline{|2}},$$

$$S_4=-\frac{(2n-2)(2n-3)(2n-4)}{\underline{|3}}.$$

Assume

$$S_r=\frac{(-1)^{r-1}(2n-2)(2n-3)\cdots(2n-r)}{\underline{|r-1}},$$

then

$$S_{r+1}=\frac{(-1)^{r-1}(2n-2)(2n-3)\cdots(2n-r)}{\underline{|r-1}}$$

$$+\frac{(-1)^{r}(2n-1)(2n-2)(2n-3)\cdots(2n-r)}{\underline{|r}}$$

$$=\frac{(-1)^{r}(2n-2)(2n-3)(2n-4)\cdots(2n-r-1)}{\underline{|r}},$$

which is of the same form as S_r. Hence,

$$S_n=\frac{(-1)^{n-1}(2n-2)(2n-3)(2n-4)\cdots[2n-(n-1)](2n-n)}{\underline{|n-1}}$$

$$=\frac{(-1)^{n-1}(2n-2)(2n-3)(2n-4)\cdots(n+1)n}{\underline{|n-1}},$$

as required.

Note. This is simply half of the expansion, $(1-1)^{2n-1}$, the other half, S_n', is equal to this but opposite in sign. $S_n+S_n'=0$.

Also solved by Horace Olson, A. M. Harding, and Geo. W. Hartwell.

409. Proposed by C. E. GITHENS, Wheeling, W. Va.

Find integral values for the edges of a rectangular parallelopiped so that its diagonal shall be rational.

SOLUTION BY WALTER C. EELLS, United States Naval Academy.

Let x, y, z be the three edges, a the diagonal, and w the hypotenuse of the right triangle of which x and y are sides. Then we seek solutions in integers of the equation $x^2 + y^2 + z^2 = a^2$.

The following method is a generalization of the well known formulas for rational right triangles; namely, if the legs be denoted by $M^2 - N^2$, and $2MN$, and the hypotenuse, therefore, by $M^2 + N^2$, then all integral values of M and N, $(M \neq N)$, will give rational integral sides.

Let $x = 2MN$, $y = M^2 - N^2$, then $w = M^2 + N^2$. This hypotenuse may be thought of under two forms:

$$A. \qquad M^2 + N^2 = m^2 - n^2, \text{ then } z = 2mn, \qquad a = m^2 + n^2;$$

$$B. \qquad M^2 + N^2 = 2m'n', \qquad \text{then } z = m'^2 - n'^2, \ a = m'^2 + n'^2.$$

Under either form it will be necessary to resolve $M^2 + N^2$ into factors, p, q, or $2m'n'$, in all possible ways.

A. $\mathbf{M^2 + N^2 = m^2 - n^2 = (m + n)(m - n) = p \cdot q}$, in which p and q are integers and $p \neq q$. For convenience take $p > q$. Then $m = (p + q)/2$ and $n = (p - q)/2$, whence p and q must be both of form (1) $2k + 1$, or both of form (2) $2k'$.

(1) *p and q both odd.* Then $M^2 + N^2$ must be odd, and M and N one even and one odd. There will be two cases to consider, according as $M^2 + N^2$ is (a) prime or (b) composite.

(a) $M^2 + N^2$ prime. The only factors are itself and unity. For a given N and M there is only one set of values p, q.

Example: $N = 1, M = 2.$ $x = 2MN = 4.$ $y = M^2 - N^2 = 3.$ $M^2 + N^2 = 5.$ $p = 5,$ $q = 1.$ $m = (p + q)/2 = 3.$ $n = (p - q)/2 = 2.$ $z = 2mn = 12.$ $a = m^2 + n^2 = 13.$ This gives the smallest rational parallelopiped, $(x, y, z, a) \equiv (4, 3, 12, 13)$.

(b) $M^2 + N^2$ composite. There are two or more sets of values of p, q.

Example: $N = 1, M = 8, x = 16, y = 63, M^2 + N^2 = 65,$

$$\begin{cases} p = 65, & q = 1; & m = 33, & n = 32; & z = 2112, & a = 2113; \\ p' = 13, & q' = 5; & m' = 9, & n' = 4; & z' = 72, & a' = 97. \end{cases}$$

This gives the two solutions: (16, 63, 2112, 2113), (16, 63, 72, 97).

Remark. It is desirable to so restrict the methods given that only relatively prime values of x, y, z, a, shall appear. Proofs for the statements given to attain this object will not be given, but they are not difficult to supply, based on simple algebra and number theory considerations.

When N and M have a common factor of the form $2k + 1$, $M^2 + N^2$ is composite [(1), (b) above], and one of the set of values p, q will have the same common factor $2k + 1$; and $(2k + 1)^2$ will be a common factor of x, y, z, a. This may be divided out giving a reduced solution with x', y', z', a' relatively prime, which will appear elsewhere in the series of relatively prime values found,

e. g., $N = 3$, $M = 6$, gives the solution (36, 27, 108, 117) by the method above, for $p = 15$, $q = 3$. Dividing by 9 there results the solution (4, 3, 12, 13) already given in (1), (a).

But *all* combinations where N and M have a common factor $2k + 1$ should

not be excluded,—only those for which in addition p and q have that factor. Thus, for $N = 3$, $M = 6$, as just given, there may be two prime solutions for $(p, q) \equiv (45, 1)$ or $(9, 5)$. They are (36, 27, 1012, 1013) and (36, 27, 26, 53).

Note. In actually computing a, it is easier not to use $a = m^2 + n^2$, but $a = 2mn + q^2$,

$$(a = m^2 + n^2 = 2mn + m^2 + n^2 - 2mn = 2mn + (m - n)^2 = 2mn + q^2),$$

since m and n need not be squared, $2mn$ is already computed, and q is usually small.

(2) *p and q both even.* Then $M^2 + N^2 = 4k$ and M and N are both even, and so have the common factor 2. It is easily shown that x, y, z, a have the common factor 2, and accordingly *no* prime solutions will be given in this case. It can be shown that the reduced forms of all these solutions appear elsewhere in prime form directly,

(a) By method A, (1) if N or M is of form $4k$, and $M^2 + N^2$ prime,

(b) One solution by A, (1) and others by B (below) if N or M is of form $4k$, and $M^2 + N^2$ composite,

(c) By method B if neither N nor M is of form $4k$.

B. $\mathbf{M^2 + N^2 = 2m'n'}$. For convenience the primes are dropped. Then $x = 2MN$, $y = M^2 - N^2$, $z = m^2 - n^2$, $a = m^2 + n^2$ where m and n $(m > n)$ are all possible factors of $(M^2 + N^2)/2$. Since $M^2 + N^2$ must be even, M and N must be (1) both odd or (2) both even.

(1) *N and M even.*

Example: $N = 2$, $M = 4$, $x = 16$, $y = 12$,

$$\begin{cases} m = 10, & n = 1, & z = 99, & a = 101, \\ m' = 5, & n' = 2, & z' = 21, & a' = 29. \end{cases}$$

Two solutions are (16, 12, 99, 101), (16, 12, 21, 29).

(2) *N and M odd.*

Example: $N = 1$, $M = 3$, $x = 6$, $y = 8$, $z = 24$, $a = 26$.

The example in (2) gives a solution having the common factor 2. It is easy to show that this is always the case when N and M are both odd. To find relatively prime solutions by method B, there should be excluded the following cases. (Details of proof omitted.)

(a) $N = 2k + 1$, $M = 2k' + 1$.

(b) $N = 2k$, $M = 2k'$, for solutions only where m and n are both even. For again it develops that x, y, z, a have the common factor 2,

e. g., $N = 4$, $M = 8$. Exclude for $(m, n) \equiv (20, 2)$ or $(10, 4)$ but retain for $(m, n) \equiv (40, 1)$.

(c) When N and M have a common factor $2k + 1$, for solutions only where m and n have the same common factor $2k + 1$,

e. g., $N = 6$, $M = 12$. Exclude solutions where $(m, n) \equiv (30, 3)$ and $(15, 6)$, but retain them where $(m, n) \equiv (90, 1)$, $(45, 2)$, $(18, 5)$, $(10, 9)$ to secure prime solutions only.

GENERAL REMARKS.

1. The methods of A and B above suffice to determine an indefinite number

of prime solutions of the required equation. Those for which a is less than 100, with the method, A or B, by which they are obtained, are as follows:

x,	y,	z,	a		x,	y,	z,	a	
(4,	3,	12,	13)	A	(12,	5,	84,	85)	A
(16,	12,	21,	29)	B	(64,	48,	39,	89)	B
(24,	32,	9,	41)	B	(16,	63,	72,	97)	A
(36,	27,	28,	53)	A	(56,	33,	72,	97)	A

2. For M and N less than 15, there are twelve groups of two solutions each for the same diagonal a,

e. g., $\begin{cases} 16,\ 63,\ 72,\ 97\ A \\ 56,\ 33,\ 72,\ 97\ A \end{cases}$ $\begin{cases} 96,\ 28,\ 75,\ 125\ B \\ 108,\ 45,\ 44,\ 125\ A \end{cases}$

$\begin{cases} 140,\ 171,\ 24420,\ 24421\ A \\ 220,\ 21,\ 24420,\ 24421\ A \end{cases}$ $\begin{cases} 40,\ 96,\ 153,\ 185\ B \\ 72,\ 135,\ 104,\ 185\ A \end{cases}$

3. If x and y are fixed values, in certain cases there are two or more parallelopipeds possible with rational diagonals, according to the number of ways $M^2 + N^2$ may be resolved into factors when composite.

Examples:

(1) $N = 1$, $M = 8$, (16, 63, 2112, 2113),
(16, 63, 72, 97).

(2) $N = 6$, $M = 12$, (144, 108, 19, 181),
(144, 108, 299, 349),
(144, 108, 2021, 2029),
(144, 108, 8099, 8101),

and if composite solutions be allowed the two additional ones

(144, 108, 189, 261),
(144, 108, 891, 909).

4. The method may be further extended to find the diagonal of the four dimensional figure (or n dimensional figure), analogous to the parallelopiped. E. g., to find solutions in integers of $x^2 + y^2 + z^2 + v^2 = a^2$. One such is $(x, y, z, v, a) \equiv (3, 4, 12, 84, 85)$. An easier one is $(a, a, a, a, 2a)$, when a is any integer.

5. For all values of M and N less than 15, there are (unless some error in computation has been made) 125 prime rational parallelopipeds. The smallest is given above. The largest is (420, 29, 88620, 88621).

The following table gives these 125 solutions, arranged according to size of a, and indicating the method by which each was derived. There are 78 solutions found by method A, and 47 by method B.

x	y	z	a		x	y	z	a	
4	3	12	13	A	48	55	2664	2665	A
16	12	21	29	B	364	27	2652	2677	A
24	32	9	41	B	40	96	2703	2705	B
36	27	28	53	A	112	180	2805	2813	B
12	5	84	85	A					
64	48	39	89	B	80	84	3363	3365	B
16	63	72	97	A	36	77	3612	3613	A
56	33	72	97	A	84	13	3612	3613	A
					240	44	3717	3725	B

x	y	z	a	
16	12	99	101	*B*
96	28	75	125	*B*
108	45	44	125	*A*
8	15	144	145	*A*
36	77	132	157	*A*
84	13	132	157	*A*
48	20	165	173	*B*
144	108	19	181	*B*
40	96	153	185	*B*
72	135	104	185	*A*
140	171	60	229	*A*
220	21	60	229	*A*
192	80	105	233	*B*
224	132	69	269	*B*
96	128	231	281	*B*
32	60	285	293	*B*
120	64	273	305	*B*
24	7	312	313	*A*
44	117	300	325	*A*
144	108	299	349	*B*
216	63	272	353	*A*
300	125	228	397	*A*
24	32	399	401	*B*
20	21	420	421	*A*
56	192	375	425	*B*
24	143	408	433	*A*
180	189	380	461	*A*
96	28	621	629	*B*
56	192	609	641	*B*
48	20	675	677	*B*
12	35	684	685	*A*
104	153	672	697	*A*
224	132	651	701	*B*
40	9	840	841	*A*
80	84	837	845	*B*
84	187	828	853	*A*
156	133	828	853	*A*
360	81	800	881	*A*
36	27	1012	1013	*A*
216	63	1000	1025	*A*
32	60	1155	1157	*B*
336	52	1131	1181	*B*
196	147	1188	1213	*A*
48	140	1365	1373	*B*
280	96	1353	1385	*B*
28	45	1404	1405	*A*
264	23	1392	1417	*A*
80	39	3960	3961	*A*

x	y	z	a	
224	132	4221	4229	*B*
120	64	4623	4625	*B*
72	65	4704	4705	*A*
20	99	5100	5101	*A*
48	140	5475	5477	*B*
60	91	5940	5941	*A*
112	15	6384	6385	*A*
96	128	6399	6401	*B*
160	36	6723	6725	*B*
108	45	6844	6845	*A*
336	52	7221	7229	*B*
44	117	7812	7813	*A*
100	75	7812	7813	*A*
144	108	8099	8101	*B*
88	105	9384	9385	*A*
56	192	9999	10001	*B*
24	143	10512	10513	*A*
144	17	10512	10513	*A*
192	80	10815	10817	*B*
140	51	11100	11101	*A*
112	180	11235	11237	*B*
72	135	11704	11705	*A*
132	85	12324	12325	*A*
120	119	14280	14281	*A*
240	44	14883	14885	*B*
52	165	14964	14965	*A*
180	19	16380	16381	*A*
224	132	16899	16901	*B*
104	153	17112	17113	*A*
176	57	17112	17113	*A*
168	95	18624	18625	*A*
28	195	19404	19405	*A*
84	187	21012	21013	*A*
156	133	21012	21013	*A*
280	96	21903	21905	*B*
140	171	24420	24421	*A*
220	21	24420	24421	*A*
216	63	25112	25113	*A*
60	221	26220	26221	*A*
208	105	27144	27145	*A*
336	52	28899	28901	*B*
120	209	29040	29041	*A*
196	147	30012	30013	*A*
180	189	34060	34061	*A*
264	23	35112	35113	*A*
260	69	36180	36181	*A*
252	115	38364	38365	*A*
240	161	41760	41761	*A*

x	y	z	a		x	y	z	a	
64	48	1599	1601	B	312	25	48984	48985	A
160	36	1677	1685	B					
60	11	1860	1861	A	308	75	50244	50245	A
					300	125	52812	52813	A
144	108	2021	2029	B					
16	63	2112	2113	A	364	27	66612	66613	A
56	33	2112	2113	A	360	81	68080	68081	A
96	28	2499	2501	B					
					420	29	88620	88621	A

Also solved in less detail by E. E. WHITFORD, C. E. FLANAGAN, and G. I. HOPKINS.

GEOMETRY.

Note. The following remark should be made in connection with the solution of Geometry **417** in the September issue:

Three planes determine *either* a point (which may be at infinity if the intersections of the planes in pairs are parallel) *or* a straight line in which the planes are concurrent (which may be at infinity if the planes are parallel). Some of the 220 points required may therefore prove to be replaced by such lines of concurrency of three planes. Whenever the word *line* is used in this solution it refers to a straight line in which three planes are concurrent.

CALCULUS.

337. Proposed by R. P. BAKER, University of Iowa.

Show that for a, b relatively prime integers,

$$\int_0^1 |\cos 2\pi ax + \cos 2\pi bx|\,dx = \frac{2}{\pi ab}\left\{\frac{a+b}{\sin\dfrac{\pi}{a+b}} - \frac{a-b}{\sin\dfrac{\pi}{a-b}}\right\}$$

or

$$= \frac{1}{\pi ab}\left\{(a+b)\cot\frac{\pi}{2(a+b)} - (a-b)\cot\frac{\pi}{2(a-b)}\right\}$$

according as a and b are both odd or one of them is even.

SOLUTION BY THE PROPOSER.

Take the first case with $a > b$.

The zeros of the integrand in the path of integration are

$$\xi_{1r} = \frac{2r-1}{2(a+b)}, \qquad r = 1, 3, 5, \cdots (a+b)$$

and

$$\xi_{2s} = \frac{2s-1}{2(a-b)}, \qquad s = 1, 3, 5, \cdots (a-b).$$

If these are $\xi_1, \xi_2, \xi_3, \cdots \xi_{2a}$ in order of magnitude the integral is

$$\frac{1}{\pi ab}\Big[b \sin 2\pi a\xi + a \sin 2\pi b\xi\Big]^{\xi_1,\ \xi_3,\ \cdots\ \xi_{2a-1}}_{\xi_2,\ \xi_4,\ \cdots\ \xi_{2a}}.$$

Consider first the sines of $2\pi b\xi_{1r}$. The rth is preceded by $r-1$ of its own set and by s of the other set where